Learning juror competence: a generalised Condorcet Jury Theorem

Jan-Willem Romeijn and David Atkinson
Department of Philosophy
University of Groningen
j.w.romeijn@rug.nl d.atkinson@rug.nl

Abstract

This paper presents a generalisation of the Condorcet Jury Theorem. All results to date assume a fixed value for the competence of jurors, or alternatively, a fixed probability distribution over the possible competences of jurors. In this paper we develop the idea that we can learn the competence of the jurors by the jury vote. We assume a uniform prior probability assignment over the competence parameter, and we adapt this assignment in the light of the jury vote. We then compute the posterior probability, conditional on the jury vote, of the hypothesis voted over. We thereby retain the central results of Condorcet, but we also show that the posterior probability depends on the size of the jury as well as on the absolute margin of the majority.

1 Introduction

In its classical formulation, the Condorcet Jury Theorem concerns collective judgement by majority voting. For example, a jury might have to decide whether it thinks that Jack is guilty or innocent of killing Jill. Every member of the voting body, in this case the jury, votes between two alternatives, of which only one is correct. The collective chooses the alternative that receives most votes. In the example each juror votes for guilt or innocence, and the verdict that receives most votes is chosen. It is assumed that consulting a single voter gives a better chance of a correct result than flipping a fair coin: the chance that a juror correctly chooses between guilt and innocence is larger than a half. The Condorcet Jury Theorem then states that if we include ever more voters in the collective, then the chance that the collective chooses the correct alternative tends to one. Put simply, the Condorcet Jury Theorem reflects the wisdom of the crowd.

The classical formulation of the theorem concerns the chances of a correct decision only. In the example, it concerns the chance of a majority vote for guilt if in actual fact Jack is guilty. As such, it does not say anything about the probability of the alternatives being true or false after all the votes have been tallied. However, under fairly general conditions we can deduce that with increasing size of the voting body, the probability associated with the true alternative tends to one as well. We can even derive a simple formula for the probability of the two alternatives, the so-called Condorcet formula. If we assume that the alternatives get equal probability before voting has taken place, and if we further assume that the chance that a voter chooses correctly is independent of which alternative happens to be true, often referred to as the symmetry of the juror competence, then the probability of the alternatives is a function only of the absolute margin between the voters for each of the alternatives: the larger the numerical difference between those who voted for Jack's guilt against those who voted innocent, the stronger the support for Jack's guilt. The size of the jury does not come into this equation.

Despite possible appearances to the contrary, the fact that the support according to the Condorcet formula is independent of the jury size is in keeping with the Condorcet Jury Theorem itself. On the assumption of either alternative being true, and using the fact that the chance for voters of choosing that alternative is larger than a half, we can derive that the absolute margin will increase at the same rate as the size of the voting body. The Condorcet formula thus yields the correct results also if we consider ever larger voting bodies. However, the absence of jury size in the Condorcet formula is not entirely uncontroversial. Say that we compare two juries, a very large one of 1000 voters which has a comparatively narrow margin of 10 in favour of guilt, and a small jury of 10 which is unanimous in its verdict of guilt. According to the Condorcet formula, both juries convey the same probability onto the alternatives of guilt and innocence. But this is often considered to be counterintuitive.

The present paper presents a generalisation of the Condorcet Jury Theorem, in which jury size plays a more prominent role. The key idea is that the relative majority of the jury, i.e., the proportion of jurors that makes up the majority, reveals something about the competence of the jurors. Effectively, the symmetry of competence is thereby dropped, and replaced by an assumption on how we may learn about competence. Most consequences of the Condorcet Jury Theorem will hold true under the generalised theorem as well. The main departures are the following.

- The probability that the jury majority verdict is incorrect is monotonically increasing in the jury size, n, if the absolute margin, Δ, remains constant.
- The probability that the jury majority verdict is incorrect tends to one-half as n tends to infinity, if Δ remains constant in this limit.
- The probability that the jury majority verdict is incorrect tends to zero as n tends to infinity, if the fractional majority, $f = \Delta/n$, tends to a nonzero positive constant in this limit.

The exclusive dependence on the absolute margin, as revealed by the Condorcet formula, is thus seen to be an artefact of idealising assumptions, and not something inherent to jury verdicts.

2 Condorcet revisited

In this section we introduce Condorcet's jury theorem; see Ladha [1992], List and Goodin [2001], List [2008], and Dietrich [2008]. We specifically present the results of List [2004], according to which the posterior probability of the hypothesis voted over, conditional on the jury vote, only depends on the absolute margin of votes in favour and against the hypothesis.

2.1 Condorcet's theorem

Let H^1 be the hypothesis that Jack murdered Jill and H^0 the hypothesis that he did not, so $\{H^0, H^1\}$ is a partition. Suppose that of a jury of n

members trying Jack, a number n_1 vote that H^1 is true, while the remaining $n_0 = n - n_1$ members vote that H^0 is true. For both $j = 0$ and $j = 1$, we assume that if H^j is in fact true, the eventuality that jury member i votes for H^j being true, denoted by V_i^j, has some fixed chance q_j, which we will call the competence of the jurors on the hypothesis H^j. For all $i, i' = 1, \ldots, n$ and $j, j' = 0, 1$, if $i' \neq i$, we set

$$p(V_i^j | H^j \cap V_{i'}^{j'}) = p(V_i^j | H^j) = q_j \,. \tag{1}$$

The left equality says that the jurors all vote independently, and the right one that they vote with fixed competences q_j, for both $j = 0, 1$. Note that the competences of the jurors with respect to H^0 and H^1 do not refer to a general ability to judge. They are specific for the hypotheses, and the competences for the hypotheses H^0 and H^1 can differ: jurors might be more accurate if H^0 is true than if H^1 is so. Finally, we assume that the competences will be greater than one half, $q_j > 1/2$, so the judgment of jury members is better than the result of tossing a fair coin.

We can now introduce Condorcet's jury theorem. Say that Jack is indeed guilty, H^1, so that the probability for any jury member to vote in favour of Jack's guilt, V_i^1, namely q_1, is greater than one-half. Now for an ever larger jury size n, consider the relative frequency of voters in favour, $f_1 = \frac{n_1}{n} = 1 - f_0$. By the law of large numbers, the difference $f_1 - q_1$ tends to 0. Because $q_1 > 1/2$, we have that the probability of a correct majority vote, $n_1 > n_0$, tends to 1 in the limit. We refer to Ladha [1993], Berend and Paroush [1998] and Dietrich [2008] for proofs of more general versions of this theorem. All these versions concern cases where the competences are sampled from a fixed distribution, thus modeling the heterogeneity of the jury.

2.2 Inverse probability

List [2004] emphasizes the importance of an inverted version of this result. Rather than calculating the probability of a majority of votes V_i^j given the truth of H^j, we want to know the probability of the correctness of the hypothesis H^j, given a majority of votes V_i^j. For convenience we denote the difference between votes for H^1 and H^0, also called the absolute margin of the majority, by $\Delta = n_1 - n_0$. We denote the entire jury vote by $V_{n\Delta} =$

$\cap_{i=1}^n V_i^{u(i)}$. Here $u(i) = 1$ if jury member i voted for H^1 and $u(i) = 0$ if she voted for H^0, so that $n_1 = \sum_i u(i)$. By Bayes' theorem we have

$$\begin{aligned} p(H^1|V_{n\Delta}) &= \frac{p(\cap_i V_i^{u(i)}|H^1)p(H^1)}{p(V_{n\Delta})} \\ &= \frac{\prod_i p(V_i^{u(i)}|H^1)p(H^1)}{p(V_{n\Delta})} \\ &= \frac{q_1^{n_1}(1-q_1)^{n_0}p(H^1)}{p(V_{n\Delta})}, \end{aligned}$$

where $p(H^1)$ is the prior probability of the hypothesis H^1. For H^0 we can derive a similar expression, replacing q_1 by q_0 and swapping the roles of n_0 and n_1. The denominator $p(V_{n\Delta}) = p(\cap_i V_i^{u(i)})$ is the same in both equations.

We can avoid calculating the denominator $p(V_{n\Delta})$ by using the posterior odds instead of the posterior probability:

$$\frac{p(H^1|V_{n\Delta})}{p(H^0|V_{n\Delta})} = \frac{q_1^{n_1}(1-q_1)^{n_0}p(H^1)}{q_0^{n_0}(1-q_0)^{n_1}p(H^0)}. \tag{2}$$

These are the odds that Jack is guilty, given the jury verdict. Since $n_0 = {}^{(n-\Delta)}/_2$ and $n_1 = {}^{(n+\Delta)}/_2$, the posterior odds depends both on the absolute margin Δ and on the jury size n.

For the posterior odds we can derive an inverse variant of Condorcet's theorem. If we let the jury size n go to infinity, and assume a fixed relative frequency $f_1 > {}^1/_2$, we can derive the exact conditions under which the posterior odds for H^1 will tend to infinity as well. We can write

$$\frac{p(H^1|V_{n\Delta})}{p(H^0|V_{n\Delta})} = \left(\frac{q_1^{f_1}(1-q_1)^{(1-f_1)}}{q_0^{(1-f_1)}(1-q_0)^{f_1}}\right)^n \frac{p(H^1)}{p(H^0)} = L^n \frac{p(H^1)}{p(H^0)}.$$

For n going to infinity, these odds will tend to infinity on the condition that $L > 1$, independently of the prior odds for H_1. Now if $\frac{1}{2} < q_1 \leq q_0 < 1$, i.e., the jurors are at least as competent at acquitting an innocent Jack as they are at condemning a guilty one, then this condition is satisfied. If, however, $\frac{1}{2} < q_0 < q_1 < 1$, i.e., the jurors are less competent at acquitting an innocent Jack than they are at condemning a guilty one, then the odds of a correct verdict tend to infinity in the infinite n limit only if f_1 is large enough.[1]

[1] The exact condition is obtained by solving f_1 for $\log L > 0$. This yields the solution that $f_1 > \frac{1}{1+x}$, in which $x = {}^{\log \frac{q_1}{1-q_0}}/_{\log \frac{q_0}{1-q_1}}$. We are not aware of earlier formulations of

2.3 Condorcet's formula

By making two further assumptions we can arrive at the result that is central to List [2004], the so-called Condorcet formula. First, we assume that the priors of H^0 and H^1 are equal, $p(H^1) = p(H^0)$, although this is not a crucial assumption. It means that initially we judge it equally probable that Jack committed the murder as that he did not. Second, and this *is* crucial for List's result, we assume that the competences of jury members with respect to H^0 and H^1 are equal, $q_0 = q_1 = q$, meaning that all jury members are precisely as reliable in condemning a murderer as they are in acquitting an innocent suspect. With these two assumptions Equation (2) simplifies to the Condorcet formula

$$\frac{p(H^1|V_{n\Delta})}{p(H^0|V_{n\Delta})} = \left(\frac{q}{1-q}\right)^{\Delta}. \tag{3}$$

Because of the requirement $q > 1/2$, we have that $\frac{q}{1-q} > 1$ so that the posterior odds that Jack killed Jill is larger than 1 if $\Delta > 0$ and smaller than 1 if $\Delta < 0$. The posterior odds depends only on the absolute margin between the numbers of correct and incorrect votes brought out by the jury members, and not on their total number. Note that this is perfectly consistent with both the Condorcet theorem and its inverse version for posterior odds: for increasing jury size n and fixed competence q_1, the expected value of Δ increases with n, and for increasing jury size n and fixed relative frequency f_1, the value of Δ increases with n.

Now focus on the fact that, for a given Δ, the posterior odds do not depend on the jury size. Imagine there are two juries, one with 10 members and one with 100 members. Suppose that both juries vote on the guilt of Jack, and that the 10-member jury unanimously votes for guilt while the 100-member jury votes by 56 in favour, and 44 against Jack's guilt. Which of the two juries then makes the guilt of Jack more probable? Well, the majority in the former is less than the majority in the latter, i.e. $\Delta_{10} = 10 < 12 = \Delta_{100}$, so that, according to Equation (3), the probability of Jack's guilt is greater for the larger than for the smaller of the two juries. Hence, if we want to have as much certainty as we can get, apparently we should prefer the verdict of the larger jury.

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this result but since it is fairly straightforward we will not provide a more detailed proof of the condition.

As has been observed in Bovens and Hartmann [2003] and in Goodin and Estlund [2004], and as we will further argue below, there is something suspect in this conclusion. Indeed, we surmise that Equation (3) is too strong an idealization: it is based on the unwarranted assumption of a symmetrical and fixed competence. In the following we provide a model in which both of these assumptions are dropped. Notably, a similar approach was taken by Goodin and Estlund, who developed a model in which the jury vote is first used to estimate the competence of the jurors, after which the estimated jury vote is employed to determine the support that the vote lends to the alternatives voted over. However, as we will explain in more detail in Section 4, their approach has some serious shortcomings.

3 A counterintuitive consequence

In this section we argue, by means of a classical statistical analysis, that there is indeed something suspect about the Condorcet formula (3), according to which only the absolute margin matters when one assesses the probability that a jury vote lends to the hypothesis voted over. The problem is that no account has been taken of how probable the jury votes are to begin with.

3.1 Intuitive characterisation

We first present the problem with the Condorcet formula in non-technical terms. As was noted in the foregoing, if two juries reach a verdict on the case of Jack and Jill, then the Condorcet formula tells us that the support each of these juries lend to their verdict is determined by the difference in the number of voters for the two alternatives voted over. Of course, if the one jury consisted of jurors who are better informed and more confident in their individual judgement than the other jury, then this might well influence the support that each of the juries confers. In particular, it may so happen that the competence of members of the one jury is higher than the competence of members of the other jury.

Now we turn to the example of the two juries presented in the foregoing. We might already feel that the smaller but unanimous jury somehow has an edge over the large and divided one. We can advance a variety of reasons for that intuition. In this paper we will develop the view that a jury vote tells us something more than just how the members in the jury think about the alternatives put to them, and that it also reveals how difficult it was for them to reach their judgement. This is relevant because, as suggested before, we should prefer a jury that is highly competent with respect to the case in question over a jury that does only marginally better than tossing a coin.

The discussion of the present section shows, by means of a classical statistical analysis, that we can in fact learn something about the competence of the jurors from the jury vote. In this section we only learn something negative, namely that the supposition that the competence of the jurors in the two juries of the example are equal cannot reasonably be maintained. This is most easily understood by looking at Figure 1, which is further explained in the caption. In the following sections, we set up a model in which we learn much more than just that.

3.2 A confidence interval for juror competence

We now make this intuitive problem with the Condorcet formula mathematically precise by constructing a so-called confidence interval for juror competence that depends on both majority and jury size. As in the previous section we assume that H^1 is true and that the competence parameter is q. Each juror votes independently and with identical probability, so that the number of votes n_1 has a binomial probability distribution. Its expectation is $\mathrm{E}[n_1] = nq$, and the standard deviation is $\mathrm{SD}[n_1] = \sqrt{nq(1-q)}$. So the mean and standard deviation of the majority Δ are

$$\begin{aligned}\mathrm{E}[\Delta] &= \mathrm{E}[n_1] - \mathrm{E}[n_0] = 2\mathrm{E}[n_1] - n = n(2q-1)\,,\\ \mathrm{SD}[\Delta] &= 2\sqrt{nq(1-q)}\,.\end{aligned}$$

For any given competence q and jury size n, we have a probability of roughly 95% that Δ lies within the specific bounds of two standard deviations around the mean, $\mathrm{E}[\Delta] - 2\mathrm{SD}[\Delta] < \Delta < \mathrm{E}[\Delta] + 2\mathrm{SD}[\Delta]$.

On this basis we can construct a confidence interval for q. Suppose Δ lies at the edge of the interval indicated above. Then we would have one or other of the following:

$$\Delta = n(2q-1) \pm 4\sqrt{nq(1-q)}\,,$$

which can be solved for q as a function of Δ, yielding the two roots

$$q_{\min}\,,\, q_{\max} = \frac{1}{2(n+4)}\left(n+4+\Delta \pm \sqrt{n+4-\frac{\Delta^2}{n}}\right) .$$

By way of interpretation, we say that any competence q inside the interval $[q_{\min}\,,\, q_{\max}]$ entails that the given majority Δ and jury size n are not so improbable that they are a cause for worry. If the $q_{\min}$ of a jury characterised by n and Δ is greater than the $q_{\max}$ of a jury with n' and Δ',

$$q_{\min}(n,\Delta) > q_{\max}(n',\Delta') ,$$

then we can say that something very improbable has occurred: at least one of the two juries has in that case voted oddly. And this should give us cause to reconsider the assumptions of the statistical model at issue. Specifically, such a result would invite us to reconsider the assumption that the competences of the jurors from the two juries are the same.

3.3 Application to the example

Now let us have a second look at the example provided in Section 2. In Figure 1 the extremal values of the competence, $q_{\min}$ and $q_{\max}$ respectively, are plotted against the majority Δ for the two juries of 10 and of 100 members. For $n = 10$ and $\Delta = 10$ it turns out that $q_{\min}(10, 10) = 0.786$, while for and $n = 100$ and $\Delta = 12$ we find $q_{\max}(100, 12) = 0.606$. With the given votes, we must therefore conclude that something highly improbable did occur. Since $q_{\max}(100, 50) = 0.783$ and $q_{\max}(100, 52) = 0.792$, we need a majority of at least 76 against 24 in the jury of 100 to feel that there is no cause for worry.

Now perhaps we simply know the numerical value of the juror competence. Or perhaps we know that all jurors have equal competence without knowing its value, in which case we might say that the competence of the jury is larger than what is suggested by the larger jury, or that the competence of the jurors is much smaller than what is suggested by the smaller jury, or possibly both. However, in any such case the result of List is applicable, and we must simply conclude that we have witnessed a freak accident.

Alternatively, we might conjecture that the two juries have different values for the juror competence. In order to sort this out, we can formulate the hypothesis that the jurors from the smaller jury are in fact more competent,

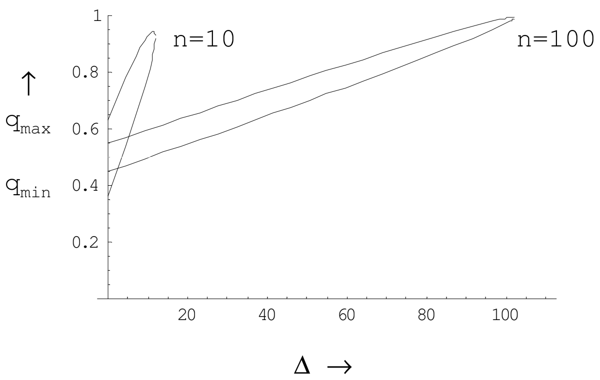

Figure 1: A graph of the bandwidth of reasonable values of the competence q against the size of the jury majority Δ, for both jury sizes $n = 10$ and $n = 100$. For each jury separately, the areas within the ellipses indicate which values of the competence can be considered reasonable, depending on the size of the majority, i.e., the absolute difference between the number of voters for the two alternatives. Along the vertical line at $\Delta = 10$, we find that the reasonable values for the competence of jurors from the jury with size 10 are all above 0.783, whereas the reasonable values for the competence of the jurors from the jury with size 100 are all below 0.606. In other words, we cannot reasonably suppose that jurors from the two juries have equal competence.

and perform a statistical test on this. But the idea that the jury vote tells us something about the competence of the jurors can also be taken a step further. We can say that the vote of the jury indicates something about the competence of the jury directly, and this knowledge may be used with advantage in the choice between jury verdicts. Specifically, the unanimous vote of the jury of 10 should perhaps weigh more heavily, despite the rule of Equation (3), simply because the unanimity suggests that the jurors are competent.

In other words, the suggestion here is that a jury vote reflects more than just the truth or falsity of the hypothesis voted over. It also conveys information on how easy it is for jurors to vote correctly. A close call in the jury, such as the small majority of 12 in the jury of 100 members, indicates that the jurors find it hard to tell whether Jack murdered Jill, while the unanimous vote of the small jury seems to suggest that the jurors find Jack's guilt fairly clear. The main contribution of this paper is in making precise what the size of the majority tells us about the competence of the jurors, and what the consequences are for assessing the jury vote.

4 Alternative explanations

Before starting on this, however, we briefly consider some alternative explanations for the jury votes considered above.

4.1 Asymmetry or heterogeneity of competence

List [2004] shows that if we do not make the assumption of symmetric competence, $q_0 = q_1$, but instead let these competences vary independently, the posterior odds do depend on the jury size. Depending on which of the two competences is larger, a larger jury with equal absolute margin will have smaller or larger posterior odds for the hypothesis. By choosing the competences in the right way, this effect may even cause the posterior odds to lean towards the minority vote. However, the model with differing but fixed competences q_0 and q_1 fails to capture the intuitions on jury verdicts voiced above. It introduces a dependence on jury size of an entirely different nature, one that is not related to our present concerns.

It may seem possible to provide a partial explanation for the above jury votes by including the heterogeneity of the jurors in the model, e.g., Ladha [1993], Berend and Paroush [1998] and Dietrich [2008]. Intuitively, if we assume a certain spread in the competence of jurors, as expressed in an assumed probability distribution over the competence parameter, then the vote from a larger jury will have added value. Just as in the case of the fixed single competence, mistakes of jury members are washed out by larger numbers. But in addition, the uncertainty stemming from randomly sampling the jurors is also diminished. Because of this added effect, it is to be expected that both the absolute majority and the jury size play a role in the eventual trust we put in the jury vote.

The exact effects of this on the confidence we have in a jury are certainly worth exploring, and in our concluding remarks we shall indicate how that might be done. However, for present purposes it is more important to emphasise that, whatever the modeling details, the inclusion of competence heterogeneity is not going to accommodate the intuitions voiced in Section 3. At first glance, there may be a striking similarity between the probability distribution over competences that expresses this heterogeneity, and the distribution over competences that we shall be using in this paper. However, the latter distribution expresses our ignorance over the value of the competence of a homogeneous jury, and this is very different from the real variability of competences within a heterogeneous jury. Accordingly, in this paper we adapt the distribution over competences in the light of the jury vote, whereas it is a fixed distribution in the models that concern heterogeneity. In this respect, the model of the present paper differs from almost all earlier models.

4.2 Uncertainty concerning competence

Goodin and Estlund [2004] are an exception. They notice a dependence between the jury vote and the competence very similar to what we noted in Section 3, and they argue that the competence of the jurors can be determined from the jury vote itself. However, the way they determine the competence is markedly different.

We describe the proposal of Goodin and Estlund in some detail. As before, it is assumed that the jury members choose between two alternatives, H^0 and H^1, that a total number n_0 chooses H^0 while a number n_1 chooses H^1, and that $n_1 > n_0$. According to Goodin and Estlund, the competence of jurors can be estimated by

$$\hat{q}_j = \frac{n_j}{n}.$$

That is, the proportion of voters that chooses a particular option is an estimate of how competent the voters are on that particular alternative. Note, however, that the proportion can be read in two different ways: it may be that H^1 is the correct alternative and that voters are competent, choosing H^1 with a majority so that $q_1 > \frac{1}{2}$. But it may also be that H^0 is the correct alternative and that voters are incompetent, choosing H^0 with a minority so that $q_0 < \frac{1}{2}$.

After providing these estimates for the competences on the basis of the vote, Goodin and Estlund discuss the support that a jury lends to the alternatives. Importantly, they show sensitivity to the fact that we do not know the competence in advance, and they take into consideration that the proportion of voters indicates this competence. This goes in the same direction as what we are proposing in the following. However, to our mind the method of estimation is far too simplistic.

The problem with their method is that from the jury vote we cannot conclude with certainty that the jury has one of the two estimated competences. Estimating the competence in the above way may lead to a good approximation of the support by a jury, especially when the jury is large. But a more accurate expression of that support must also incorporate the epistemic uncertainty that surrounds the competence. An increase in jury size will lead to more accurate estimations, but in the model of Goodin and Estlund this fact cannot be taken into account. The analysis of the present paper fares much better in this respect.

4.3 Coherent voting

We might consider an extension of the model of jury decisions in an entirely different direction. Arguably, a small jury has a completely different group dynamics than a larger jury, and the jury verdict may reflect how the jurors have interacted. For example, it may be that jurors adjust their views to coincide with those of jurors sitting in close proximity to them. In a jury of 100 the jurors may then still be treated as approximately independent. But in a jury of 10, all jurors are in close proximity to each other. Therefore a unanimous vote in a small jury may very well be the result of mindless groupthink rather than of high juror competence. Such failures of independence will generally put the results of a jury vote in a different perspective.

In a similar vein, we may think that the coherence of the jurors in the smaller jury is indicative of the veracity of the jury verdict. This idea is at the basis of the discussion that Bovens and Hartmann [2004, section 3.6] give of the Condorcet formula. They also note its counterintuitive consequences, adapt the model to include a positive correlation between the votes, and then show that in this model a smaller unanimous jury lends more credibility to the jury verdict than a larger jury whose verdict is divided, even when the absolute margins in both juries are equal. Moreover, by adapting the parameters in the model they can vary the degree to which the coherence of jurors adds to the credibility of the verdict.

The coherence model of Bovens and Hartmann provides a successful explication of some such intuitions concerning jury votes. It sensibly drops the assumption of the independence of the jurors, and employs the truth-conduciveness of the coherence of votes to avoid the counterintuitive consequence of the Condorcet formula. A drawback of this solution is that it relies on particular parameter values that must be filled in at the start. Given these parameters, we can deduce the dependence of the posterior odds on the jury size, but this dependence is in a sense put in by hand. Yet this drawback does not mean that we should discard the coherence model completely.

Accordingly, we do not motivate the model of the following sections by a claim that it captures our intuitions on jury votes better. Rather it captures another intuition about jury votes, differing from those captured in the coherence model of Bovens and Hartmann [2004], and similar to those voiced by Goodin and Estlund [2004]. It is the idea that the competence of jurors can be partially revealed by the jury vote, as was indicated by the classical statistical analysis of Section 3. We think that this idea is of interest in its own right, and that it merits a more extensive treatment than was given by Goodin and Estlund.

5 Jury vote with unknown competence

In the following we present a model in which the jury vote is indicative of how competent the jurors are concerning the hypothesis at hand. We retain the assumption that the jurors vote independently and concentrate on relaxing the assumption of single-valued juror competences. To do this we employ Bayesian statistical inference. We first compute a posterior probability assignment over the competences q_0 and q_1 for H^0 and H^1 respectively, based on the given jury vote and a prior probability over competences and hypotheses. This inference determines how the jury vote informs us of the competence: we may derive an expectation value for the juror competences from it. More importantly, we compute the probability that a jury vote gives to the hypothesis voted over.

5.1 Learning juror competence

Before we make the model and its consequences formally explicit, we give an informal characterisation of it. Readers who have no interest in the mathematical details may skip Subsection 5.2.

In Section 3 we argued that a jury vote might reflect more than just the truth or falsity of the hypothesis voted over: the vote also conveys information on how easy it is for jurors to vote correctly. A close call in the jury, such as the small majority of 12 in the jury of 100 members, might indicate that the jurors find it hard to tell whether Jack murdered Jill, while the unanimous vote of the small jury of 10 seems to suggest that the jurors find Jack's guilt all too clear. The model of this paper aims to make precise what the relative size of the majority tells us about the competence of the jurors, and what the consequences are for assessing the jury vote.

We consider two alternatives and a jury voting between them, and we assume that one of them is correct. For each of the alternatives we assume the voters have some competence, meaning that there is some fixed chance that, if that alternative is correct, the voter chooses for that alternative. We assume that jury members are at least as good at choosing correctly as the toss of a fair coin. The crucial difference with earlier models in the literature on the Condorcet Jury Theorem is that at the start of the jury vote, we are ignorant of the juror competence: every value of the competence between $\frac{1}{2}$ and 1 is deemed possible, and is assigned some probability. We do not assume some specific value for the competence.

The use of a probability assignment over possible values of the competence allows us to express the impact of a particular jury vote on our estimations of juror competence. In fact it allows us much more detail in what we learn from the jury vote than if we merely produce a best estimate, as proposed by Goodin and Estlund [2004]. The model that is introduced below presents us with a probability assignment over possible competences after the jury vote, telling us what value of the competence is most probable, and by the shape of the probability assignment over competences, it also tells us how reliable that estimate is. Moreover, based on this probability assignment over the possible values of juror competence we can determine the support that the jury vote lends to the two alternatives voted over. In sum, we provide a probability for the two alternatives that takes into account both the jury vote itself and the fact that the jury vote tells us something about how competent the jurors are.

5.2 Mathematical model

Computing the expectation value for the jury competence is a tricky business. In the foregoing we had a partition of two hypotheses, H^0 and H^1. But since the competence parameter is unknown, we must split these hypotheses up into continuous ranges of hypotheses, H_{q_0} and H_{q_1}. The expressions $p(H_{q_j})$ should therefore be regarded as probability densities rather than themselves probabilities. The hypotheses H^0 and H^1 each consist of a range of statistical hypotheses, parameterised by q_0 and q_1 respectively. These hypotheses have the likelihoods

$$p(V_j^i | H_{q_j} \cap V_{j'}^{i'}) = q_j$$

for $j = 0, 1$. The probabilities of the aggregate hypotheses H^0 and H^1 are

$$\begin{aligned} p(H^0) &= \int_0^1 p(H_{q_0})\, dq_0 \,, \\ p(H^1) &= \int_0^1 p(H_{q_1})\, dq_1 \,. \end{aligned}$$

Further, we assume that the prior is equal and uniform over the interval $(\frac{1}{2}, 1]$, for both q_0 and q_1, meaning that $p(H_{q_j}) = 1$ for $\frac{1}{2} < q_j \leq 1$, and $p(H_{q_j}) = 0$ for $0 \leq q_j \leq \frac{1}{2}$. Thus the only prior assumption is that the jury members are not incompetent, but aside from that the prior density is flat, as an expression of the fact that prima facie we consider each competence value in the interval $(\frac{1}{2}, 1]$ equally probable. The above considerations entail

$$p(H^0) = \int_{1/2}^1 1\, dq_0 = \frac{1}{2} = \int_{1/2}^1 1\, dq_1 = p(H^1).$$

For reasons of simplicity we will not deviate from this assumption in what follows.

It is convenient to reduce the number of parameters in this statistical model to a single one by a suitable substitution of the parameters over the domain. Note that in the above setup, the use of the two parameters q_0 and q_1 does not mean that the statistical model is two-dimensional. The likelihoods involve q_0 if H^0 is true and q_1 if H^1 is true, but these are mutually exclusive hypotheses, so there is no overlap in which the likelihoods involve both parameters. Because of this we can employ a single range of hypotheses

H_r with the parameter domain $r \in [0,1]$, which is formally equivalent to the combination of q_0 in H^0 and q_1 in H^1.

Let us make this formal equivalence precise.[2] First, within the domain $r \in [0, 1/2)$, and setting $q_0 = 1 - r$, we have the following equalities:

$$\begin{array}{rcccccl} p(V_i^0|H_{q_0}) & = & q_0 & = & 1-r & = & p(V_i^0|H_r) \\ p(V_i^1|H_{q_0}) & = & 1-q_0 & = & r & = & p(V_i^1|H_r)\,. \end{array} \tag{4}$$

Similarly, in the domain $r \in (1/2, 1]$, and setting $q_1 = r$, we have the following equalities:

$$\begin{array}{rcccccl} p(V_i^0|H_{q_1}) & = & 1-q_1 & = & 1-r & = & p(V_i^0|H_r) \\ p(V_i^1|H_{q_1}) & = & q_1 & = & r & = & p(V_i^1|H_r)\,. \end{array} \tag{5}$$

In words, there is a formal equivalence between the likelihoods of the hypotheses H_r for $r < 1/2$, and those of H_{q_0} for $q_0 > 1/2$. Similarly, there is an equivalence between the likelihoods of the hypotheses H_r for $r > 1/2$ and those of H_{q_1} for $q_1 > 1/2$.

Now consider the resulting likelihoods for the hypotheses H_r. From the right hand side of Equations (4) and (5) we can see that, over the entire domain $r \in [0, 1]$, the hypotheses H_r have the likelihoods

$$\begin{aligned} p(V_i^0|H_r) &= 1-r\,, \\ p(V_i^1|H_r) &= r\,. \end{aligned}$$

By updating the separate hypotheses H_r according to these likelihoods, we are effectively updating the hypotheses H_{q_0} and H_{q_1} for each of the values $q_0 \in (1/2, 1]$ and $q_1 \in (1/2, 1]$.

Next we consider the priors over the hypotheses H_{q_0} and H_{q_1}. Recall that we assumed a uniform prior probability distribution over both of them. But we can rewrite these priors in terms of priors over the hypotheses H_r in the respective domains $r \in [0, 1/2)$ and $r \in (1/2, 1]$, as follows:

$$\begin{aligned} p(H^0) &= \int_{1/2}^{1} p(H_{q_0})\, dq_0 = \int_0^{1/2} p(H_r)\, dr\,, \\ p(H^1) &= \int_{1/2}^{1} p(H_{q_1})\, dq_1 = \int_{1/2}^{1} p(H_r)\, dr\,. \end{aligned}$$

[2] In the following we leave out the value $r = 1/2$. Since this is a measure zero event, its omission has no consequences.

Hence the uniform priors over the hypotheses H_{q_0} and H_{q_1} translate into a single uniform prior over the hypotheses H_r with $r \in [0,1]$. We can interpret the probability of H_r with $r < 1/2$ as the probability of H_{q_0} by the translation $q_0 = 1 - r$, and similarly, we can interpret the probability of H_r with $r > 1/2$ as the probability of H_{q_1} by the translation $q_1 = r$. We note, as an aside, that it is attractive to start out with uniform priors over the hypotheses H_{q_j}, or at least with priors that combine into a Beta distribution over H_r. Priors over H_{q_j} that have a different shape do not necessarily lead to posterior distributions that can be expressed in terms of a canonical function.

The substitution above is useful because we have thereby disposed of a parameter, replacing q_0 and q_1 by the single parameter r. Moreover, we can model the impact of the jury vote on the combined uniform probability assignments over $q_0 \in (1/2, 1]$ and $q_1 \in [1/2, 1]$ by modeling its impact on the uniform probability assignment over $r \in [0,1]$.

We shall condition this distribution on the jury vote $V_{n\Delta}$, characterised by the numbers of votes n_0 for H^0 and n_1 for H^1, or equivalently, by the size of the jury $n = n_1 + n_0$ and the majority $\Delta = n_1 - n_0$. Then the posterior probability distribution over H_r results in a well-known form for the posterior distribution, the Beta distribution,

$$p(H_r|V_{n\Delta}) = \frac{(n+1)!}{n_0! n_1!} r^{n_1}(1-r)^{n_0},$$

with $r \in [0,1]$. For $r > 1/2$ we are thereby indirectly specifying the posterior probability distribution over the hypotheses H_{q_1} according to the transformation $q_1 = r$, while for $r < 1/2$ we are indirectly specifying the posterior for the hypotheses H_{q_0}, using the transformation $q_0 = 1 - r$.

From this expression we can derive the posterior probability of the hypotheses H^0 and H^1:

$$p(H^0|V_{n\Delta}) = \frac{(n+1)!}{n_0! n_1!} \int_0^{1/2} r^{n_1}(1-r)^{n_0}\, dr = 1 - p(H^1|V_{n\Delta}). \qquad (6)$$

This can be written in terms of the jury size n and the majority Δ, using $n_0 = (n-\Delta)/2$ and $n_1 = (n+\Delta)/2$. The expectation values for the competences of the jurors, finally, are given by the following normalised integrals:

$$\begin{aligned}
\mathrm{E}[q_0] &= \frac{1}{p(H^0|V_{n\Delta})} \int_0^{1/2} r^{n_1}(1-r)^{n_0+1}\, dr, \\
\mathrm{E}[q_1] &= \frac{1}{p(H^1|V_{n\Delta})} \int_{1/2}^{1} r^{n_1+1}(1-r)^{n_0}\, dr.
\end{aligned}$$

If $n_1 > n_0$, then we will have that $\mathrm{E}[q_1] > \mathrm{E}[q_0]$, because on the assumption that H^0 is true a majority for H^1 is more likely if the competence q_0 is low.

Before we investigate the expression (6) in the next section, we want to address a possible criticism of the derivation of the posteriors. It may be objected that the jury vote is used twice: once for the determination of the posterior over competences, and then again for the determination of a posterior for the hypotheses based on some expected competence. But in the model above there is no such double usage. We only employ the vote to determine a probability distribution over the parameter r, which summarises the two competences q_0 and q_1. All the other probability assignments are derived from this distribution without using the data again.

6 Calculating the Posterior Probability

In the preceding section we derived a probability assignment for the hypotheses concerning Jack's guilt, under the assumption of the independence of jurors but without assuming a uniform value for the competences q_0 and q_1. The assignment is an expression in which both the size of the majority and the jury size play a role. In contrast, in Section 2 we presented the result of List [2004] that, on the assumption of any particular competence $q_0 = q_1 = q$, the probability of the hypotheses only depends on the majority.

In this section we investigate the implications of Equation (6) both analytically and numerically. The implications of the present model are in accordance with our intuitions on the relation between jury votes and the hypothesis voted over: if, in a large jury, we have a close call between the votes for the two alternatives, this is taken as a sign that it is hard to decide between the hypotheses, i.e. that the competence of the jurors is therefore probably low, and accordingly it is taken as a reason to put less trust in how the jury has voted than the trust suggested by the absolute margin alone.

We thereby provide a more accurate model of an aspect to jury voting that was initially noticed by Goodin and Estlund [2004].

6.1 Analytic results

We first give an analytic characterisation of how the probability for the hypotheses depends on jury size and absolute margin. Significantly, we retain an important consequence of the Condorcet formula, as discussed in Section 2. On the assumption that $n_1 > n_0$, or $\Delta > 0$, we have the pairwise inequality

$$p(H_r|V_{n\Delta}) \; > \; p(H_{1-r}|V_{n\Delta})$$

for all $r \in (1/2, 1]$. Hence we also have that

$$\int_{1/2}^{1} p(H_r|V_{n\Delta})\, dr \; > \; \int_{1/2}^{1} p(H_{1-r}|V_{n\Delta})\, dr \, .$$

Via the translation $1 - r = q_0$ and $r = q_1$ within $r \in (1/2, 1]$, we thus obtain the inequality $p(H^1|V_{n\Delta}) > p(H^0|V_{n\Delta})$ on condition that $\Delta > 0$.

Further, let us look at the so-called marginal likelihoods of the hypotheses H_j on the two votes $V^0_{n+1} \cap V^1_{n+2}$. These two votes effectively enlarge the jury while keeping the absolute margin Δ fixed. In Appendix A it is shown that if $\Delta = n_1 - n_0 > 0$, we have

$$p(V^0_{n+1} \cap V^1_{n+2}|H^0 \cap V_{n\Delta}) \; > \; p(V^0_{n+1} \cap V^1_{n+2}|H^1 \cap V_{n\Delta}). \tag{7}$$

Since we also have

$$\frac{p(H^1|V_{n+2,\Delta})}{p(H^0|V_{n+2,\Delta})} = \frac{p(V^0_{n+1} \cap V^1_{n+2}|H^1 \cap V_{n\Delta})}{p(V^0_{n+1} \cap V^1_{n+2}|H^0 \cap V_{n\Delta})} \times \frac{p(H^1|V_{n\Delta})}{p(H^0|V_{n\Delta})} \, ,$$

the inequality of Equation (7) entails that the odds of Jack's guilt decreases monotonically as the jury size n is increased if we hold the absolute margin $\Delta > 0$ fixed. This is in accordance with the intuitions voiced in the preceding sections, in particular that the jury size affects the probability of the hypothesis.

Secondly, we investigate the limiting behavior in order to arrive at a generalisation of Condorcet's theorem for posterior odds. The mere fact that $p(H^0|V_{n\Delta})$ increases with the jury size for fixed $\Delta > 0$ does not yet determine the limiting value for $p(H^0|V_{n\Delta})$ as n goes to infinity. However, as shown in Appendix B, if the absolute margin Δ is held constant in the limit, we find the asymptotic behavior

$$\lim_{n\to\infty} p(H^0|V_{n\Delta}) = \frac{1}{2}.$$

It is further shown that if the fractional majority, $f = \Delta/n$, is held constant in the limit, we have instead

$$\lim_{n\to\infty} p(H^0|V_{n,nf}) = 0.$$

Finally, we show that for the latter limiting behavior, it is not necessary for the majority to increase linearly with n. It is enough if Δ increases more quickly than $\sqrt{n}$ to ensure that $p(H^0|V_{n\Delta})$ tends to zero.

6.2 Numerical results

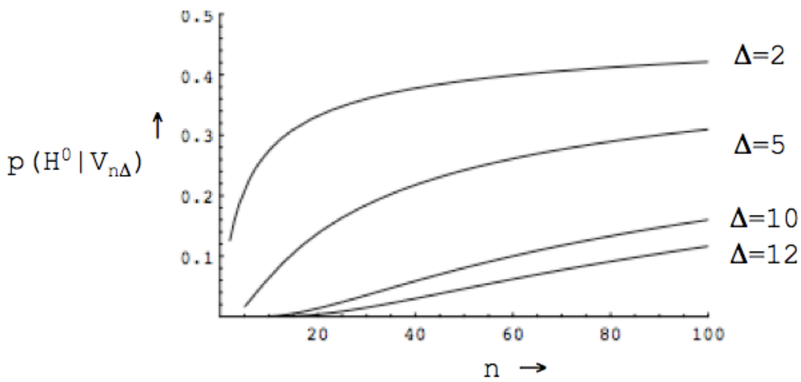

Figure 2: Graphs showing the probability of the alternative H^0 after incorporating the jury vote, $p(H^0|V_{n\Delta})$, as a function of the jury size, n, for a range of values of the absolute difference between the number of voters for the two alternatives, $\Delta = 2, 5, 10$, and 12. In all these cases the majority of the jury voted for alternative H^1. The salient aspect of the graphs is that for a fixed value of the absolute majority, the probability of the alternative supported by the minority, H^0, increases with the size of the jury. The intuitive reason for this is that, for fixed majority size, a larger jury size signals a lower juror competence.

In addition to these qualitative results, we have also done some numerical calculations with the aid of Mathematica®.

The integral in Equation (6) can be written as a known transcendental function, the so-called regularized incomplete Beta function. In Figure 2 we have plotted the relation between the probability $p(H^0|V_{n\Delta})$ and

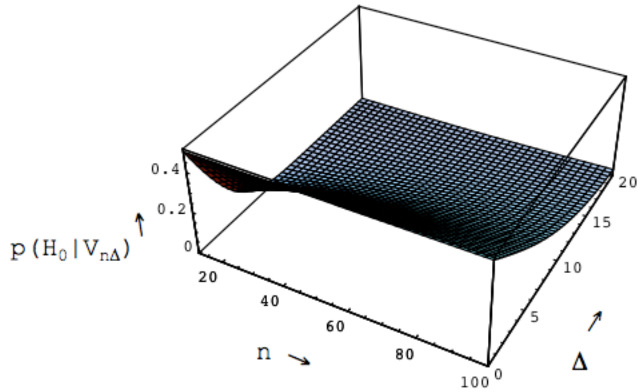

Figure 3: A graph showing the probability of the alternative H^0 after incorporating the jury vote, $p(H^0|V_{n\Delta})$, as a function of the jury size n and the size of the majority Δ. As in Figure 2, we see that for any majority size, the probability of H^0 increases.

the jury size $n = n_1 + n_0$, for various values of the size of the majority $\Delta = n_1 - n_0$. These calculations illuminate the case of the two juries considered at the beginning of this paper. With a unanimous verdict of guilt in a jury of 10, $\Delta = 10$ and $n = 10$, for example, we find the probability $p(H^1|V_{10,10}) = 0.9995$. For a jury of 100 with a majority of 12 for guilt, on the other hand, we calculate a smaller probability that the jury verdict is correct, namely $p(H^1|V_{100,12}) = 0.8839$. The important point here is that the probability depends not only on the majority, as it did when we chose some fixed competence. It decreases as n is increased, and this effect may counterbalance a difference in the size of the majority.

In Figure 3 we see the dependence of the probability $p(H^0|V_{n\Delta})$ on the majority size Δ spelled out in more detail. Note first that for $\Delta = 0$, this probability is one-half, as it should be. Furthermore, for any fixed jury size, the probability of the hypothesis H^0 decreases with increasing majority size. And finally, for fixed majority size Δ and increasing jury size n, we can see that the probability $p(H^0|V_{n\Delta})$ slowly increases towards a half again. For very large juries, as also suggested by Figure 2, a small majority does not carry much weight.

7 Concluding remarks

Now that we have obtained these results, what can we say of earlier results in the literature? Recall that the model of Goodin and Estlund [2004] is motivated by the same intuition as the model of this paper. However, we claim that the present model is more convincing than the one that Goodin and Estlund provide, because it takes into account both the learning of juror competence, and the uncertainty associated with this learning. That is, unlike Goodin and Estlund, our model accommodates the fact that the juror competence may not be reflected exactly in the proportion of jurors. We think this is an improvement over their model.

The main result of List [2004] was that the probability of the hypotheses voted over only depends on the absolute margin Δ. Of course, this is still a valid point under the assumption of fixed symmetric competence. But with the foregoing considerations in mind, we see that if we do not know the competences q_0 and q_1, and if we decide to learn about these competences on the basis of the jury vote, then both the absolute majority and the jury size matter. The model of this paper thus provides an alternative account of how the confidence we may have in a jury vote depends on the size of the jury as well as on the absolute majority, besides the explanation provided by Bovens and Hartmann [2003].

We conclude with some suggestions on how to develop the results of the present paper. First, we think that it is important for the practical applicability of Condorcet-style results to relax the assumption of the theorems concerning the independence of the jurors. As mentioned above, Bovens and Hartmann successfully model a jury vote with dependent jurors, and it will be interesting to see if that model can be combined with the model presented in this paper. Another way to incorporate the jury dynamics into the analysis is presented by adapting the prior, so as to make it less sensitive to almost or entirely unanimous voting. As indicated before, we might think that unanimity in small juries is due to mindless groupthink, and not a sign of a high juror competence or of truth-conducive coherence. If so, we can correct for a possible overestimation of the competence by choosing a prior over competences that is peaked around $r = 1/2$.

Perhaps a more accurate way of modeling the interaction between group members is by dropping the assumption on the likelihoods that is expressed in the left-hand equality of Equation (1) altogether. In the model presented in this paper, jury votes $V_{n\Delta}$ are characterised by the numbers n and Δ only, because the likelihoods of the hypotheses H^j only depend on these numbers. But we might also partition the space of possible jury votes differently, according to other characteristics of the votes, and employ hypotheses that have more complicated likelihood functions over that space. While this will no doubt provide interesting new insights, we can scarcely hope to attain analytic results for a model with these more involved statistical hypotheses.

An entirely different line of research concerns the real variation of competences within the jury, as discussed earlier in this paper. For example, Dietrich [2008] shows that the classical Condorcet jury theorem still holds if we suppose that the competences of jurors vary, as long as their average competence is larger than $1/2$. This raises the question whether we can also derive an expression for the posterior odds of the hypothesis on the assumption of a certain spread in the competences of the respective jurors. A suitable statistical setting for answering this question is hierarchical Bayesian modeling, in which we may suppose the juror relative competence q_{ij} to be drawn at random from a distribution of possible values for the competence. Again, analytic results may be very hard to come by, but software packages such as WinBUGS® are well equipped to investigate such models numerically.

Finally, we expect that much can be gained by applying the present insights to the discussion over the coherence measures proposed in Bovens and Hartmann [2004], and continued in Haenni and Hartmann [2006]. The reliability parameter employed there is formally similar to the competence parameter employed in the present paper. It will be interesting to see if and how we can adapt our estimations of the reliability of measurement apparatuses or witnesses from their coherence. Because the well-known impossibility result of Bovens and Hartmann relies on variability in the reliability parameter, and because in the present paper we have shown how to adapt the probability assignment over values of this parameter, we expect that casting their impossibility result in terms of the present findings will lead to interesting new insights.

Acknowledgements

We would like to thank the *Grundlegung* reading group of the Philosophy Faculty of the University of Groningen and the audiences at the BSPS 2008 conference in St Andrews and the FMP 2008 conference in Tilburg for helpful discussion. This research was carried out as part of a project funded by the Dutch Organization of Scientific Research (NWO VENI-grant nr. 275-20-013). We also thank the Spanish Ministry of Science and Innovation (Research project FFI2008-1169) for generous support.

A Relative size of the marginal likelihoods

In this appendix we prove that the marginal likelihood of the hypothesis H^0 for the combined votes $V^0_{n+1} \cap V^1_{n+2}$, given an earlier jury vote $V_{n\Delta}$ for which $\Delta > 0$, is larger than the corresponding likelihood of the hypothesis H^1. Mathematically,

$$p(V^0_{n+1} \cap V^1_{n+2} | H^0 \cap V_{n\Delta}) \;>\; p(V^0_{n+1} \cap V^1_{n+2} | H^1 \cap V_{n\Delta}). \tag{8}$$

We will do so by writing out the marginal likelihoods in terms of the likelihoods of the statistical hypotheses H_r for $r < 1/2$ and $r \geq 1/2$ respectively.

We first determine the likelihoods of the hypotheses H_r for the two votes $V^0_{n+1} \cap V^1_{n+2}$:

$$p(V^0_{n+1} \cap V^1_{n+2} | H_r \cap V_{n\Delta}) = r(1-r).$$

Recall that the hypotheses H^0 and H^1 are composed of the statistical hypotheses H_r. The likelihood of the hypothesis H^0 for the two votes $V^0_{n+1} \cap V^1_{n+2}$ is

$$\begin{aligned} & p(V^0_{n+1} \cap V^1_{n+2} | H^0 \cap V_{n\Delta}) \\ & \quad = \int_0^{1/2} p(H_r | H^0 \cap V_{n\Delta}) p(V^0_{n+1} \cap V^1_{n+2} | H_r \cap H^0 \cap V_{n\Delta})\, dr \\ & \quad = \frac{1}{p(H^0 | V_{n\Delta})} \frac{(n+1)!}{n_0! n_1!} \int_0^{1/2} r^{n_1} (1-r)^{n_0}\, r(1-r)\, dr\,. \end{aligned} \tag{9}$$

in which we have used the normalisation $p(H^0|V_{n\Delta})$. Following Equation (6), we have within $r \in [0, 1/2]$ that

$$
\begin{aligned}
p(H_r|H^0 \cap V_{n\Delta}) &= \frac{p(H_r \cap H^0|V_{n\Delta})}{p(H^0|V_{n\Delta})} = \frac{p(H_r|V_{n\Delta})}{p(H^0|V_{n\Delta})} \\
&= \frac{1}{p(H^0|V_{n\Delta})} \frac{(n+1)!}{n_0!n_1!} r^{n_1}(1-r)^{n_0} \,. \qquad (10)
\end{aligned}
$$

The marginal likelihood of the hypothesis H^1 for the two votes is given by a similar expression, with the difference that the integration bounds are $1/2$ and 1, and that the normalisation is $p(H^1|V_{n\Delta})$.

In order to compare the two marginal likelihoods, it will be convenient to write Equation (10) in terms of the same integration bounds, making use of the symmetry in the integral expression:

$$
\begin{aligned}
&p(V^0_{n+1} \cap V^1_{n+2}|H^1 \cap V_{n\Delta}) \\
&\quad = \frac{1}{p(H^1|V_{n\Delta})} \frac{(n+1)!}{n_0!n_1!} \int_{1/2}^{1} r^{n_1}(1-r)^{n_0}\, r(1-r)\, dr \\
&\quad = \frac{1}{p(H^1|V_{n\Delta})} \frac{(n+1)!}{n_0!n_1!} \int_{0}^{1/2} r^{n_0}(1-r)^{n_1}\, r(1-r)\, dr \,. \qquad (11)
\end{aligned}
$$

We can now compare the two marginal likelihoods by comparing the functions appearing under the integration sign. Specifically, we will investigate the expression

$$
\begin{aligned}
&p(V^0_{n+1} \cap V^1_{n+2}|H^0 \cap V_{n\Delta}) - p(V^0_{n+1} \cap V^1_{n+2}|H^1 \cap V_{n\Delta}) \\
&\quad = \int_0^{1/2} \big(p(H_r|H^0 \cap V_{n\Delta}) - p(H_r|H^1 \cap V_{n\Delta})\big)\, r(1-r)\, dr \\
&\quad = \frac{(n+1)!}{n_0!n_1!} \int_0^{1/2} \left(\frac{r^{n_1}(1-r)^{n_0}}{p(H^0|V_{n\Delta})} - \frac{r^{n_0}(1-r)^{n_1}}{p(H^1|V_{n\Delta})} \right) r(1-r)\, dr \,, \qquad (12)
\end{aligned}
$$

the difference between the marginal likelihoods of Equations (9) and (11). If this function is positive, then the marginal likelihood of H^0 is larger than that of H^1, which is what we have set out to prove.

The expression inside the integral of Equation (12) consists of two parts. We now make some observations on the part between brackets,

$$
g(n_0, n_1, r) = \frac{r^{n_1}(1-r)^{n_0}}{p(H^0|V_{n\Delta})} - \frac{r^{n_0}(1-r)^{n_1}}{p(H^1|V_{n\Delta})}.
$$

First, because of the normalisations, $p(H^j|V_{n\Delta})$, we have

$$
\int_0^{1/2} g(n_0, n_1, r)\, dr = 0. \qquad (13)
$$

Next, if we assume that $n_1 > n_0$, Equation (7) says that $p(H^1|V_{n\Delta}) > p(H^0|V_{n\Delta})$, so that we have

$$g(n_0, n_1, 1/2) = \left(\frac{1}{p(H^0|V_{n\Delta})} - \frac{1}{p(H^1|V_{n\Delta})}\right)\frac{1}{2^n} > 0\,. \tag{14}$$

Furthermore, the equation $g(n_0, n_1, r) = 0$ has two solutions in r. One is $r = 0$, the other is

$$r^* = \frac{c}{1+c} \qquad \text{with} \qquad c = \left(\frac{p(H^0|V_{n\Delta})}{p(H^1|V_{n\Delta})}\right)^{1/\Delta}\,. \tag{15}$$

Together with Equations (12) and (14), Equation (15) entails that in the domain $r \in (0, r^*)$ we have that $g(n_0, n_1, r) < 0$ while in $r \in (r^*, 1/2]$ we have that $g(n_0, n_1, r) > 0$. Finally, with Equation (13) this entails that

$$-\int_0^{r^*} g(n_0, n_1, r)\, dr = \int_{r^*}^{1/2} g(n_0, n_1, r)\, dr. \tag{16}$$

In other words, the entire negative contribution to the integral of Equation (13) lies in $r < r^*$, while the entire positive contribution to it lies in $r > r^*$. All this is illustrated in Figure 4.

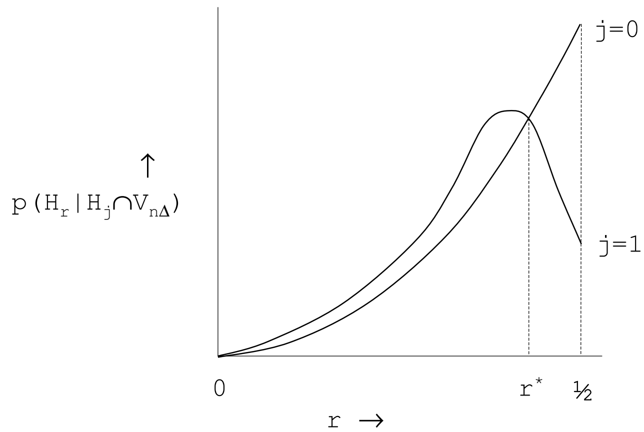

Figure 4: Graphs of the functions $p(H_r|H^j \cap V_{n\Delta})$ for $j = 0, 1$ against $r \in [0, 1/2]$. The values of n and Δ are kept fixed. As expressed in Equation (16), the two areas in between the two curves are equal.

We make one further observation on the function $r(1-r)$, namely that it is monotonically increasing in r over the domain $r \in [0, 1/2]$. Now recall that in the domain $r \in [0, r^*]$, the contribution of the integral is entirely negative.

The factor with which the function $g(n_0, n_1, r)$ is multiplied over this domain is, on average, strictly less than $r^*(1-r^*)$, and the contribution to the whole integral of Equation (12) therefore has the following lower bound:

$$\int_0^{r^*} g(n_0, n_1, r)\, r(1-r)\, dr \ > \ r^*(1-r^*) \int_0^{r^*} g(n_0, n_1, r)\, dr. \tag{17}$$

In the domain $r \in [r^*, 1/2]$, on the other hand, the integral is entirely positive, and the factor with which the function $g(n_0, n_1, r)$ is multiplied is, on average, strictly more than $r^*(1-r^*)$, thus leading to a contribution with a lower bound

$$\int_{r^*}^{1/2} g(n_0, n_1, r)\, r(1-r)\, dr \ > \ r^*(1-r^*) \int_{r^*}^{1/2} g(n_0, n_1, r)\, dr. \tag{18}$$

Combining these two equations, we have a lower bound of the difference between the two marginal likelihoods covering the entire domain of r.

Hence we can determine the lower bound of the difference between the marginal likelihoods, as follows:

$$\begin{aligned}
&p(V^0_{n+1} \cap V^1_{n+2} | H^0 \cap V_{n\Delta}) - p(V^0_{n+1} \cap V^1_{n+2} | H^1 \cap V_{n\Delta}) \\
&\quad = \int_0^{1/2} g(n_0, n_1, r)\, r(1-r)\, dr \\
&\quad = \int_0^{r^*} g(n_0, n_1, r)\, r(1-r)\, dr + \int_{r^*}^{1/2} g(n_0, n_1, r)\, r(1-r)\, dr \\
&\quad > r^*(1-r^*) \left(\int_0^{r^*} g(n_0, n_1, r)\, dr + \int_{r^*}^{1/2} g(n_0, n_1, r)\, dr \right) = 0.
\end{aligned}$$

The crucial step in this derivation is of course the inequality, which is based on the two lower bounds of Equations (17) and (18). Together they establish Equation (8).

B Limiting behaviour of the probabilities

The posterior probability for H^0 conditional on the jury vote $V_{n\Delta}$ can be written as a regularized incomplete Beta function; see Abramowitz and Stegun [1964, p. 263], formulae (6.6.1) and (6.6.2). Specifically,

$$\begin{aligned} p(H^0|V_{n\Delta}) &= \frac{(n+1)!}{n_0!n_1!}\int_0^{\frac{1}{2}} dr\, r^{n_1}(1-r)^{n_0} \\ &= \frac{B_{\frac{1}{2}}(n_1+1, n_0+1)}{B(n_1+1, n_0+1)} \\ &\equiv I_{\frac{1}{2}}(n_1+1, n_0+1)\,. \end{aligned} \tag{19}$$

In this appendix we exploit certain asymptotic properties of the regularized incomplete Beta function to show that $p(H^0|V_{n\Delta})$ tends to one-half as n tends to infinity at constant Δ, but to zero if the fractional majority, $f = \Delta/n$, is held constant in the limit. It is also shown that if Δ increases more quickly than $\sqrt{n}$, $p(H^0|V_{n\Delta})$ still tends to zero in the limit of n to infinity.

Theorem 1

If $\Delta = n_1 - n_0 \geq 0$ is constant, then $I_{\frac{1}{2}}(n_1+1; n_0+1)$ tends to $1/2$ in the limit that $n = n_1 + n_0$ tends to infinity.

Proof

On changing the integration variable from r to $t = (1-2r)^2$, we find

$$I_{\frac{1}{2}}(n_1+1; n_0+1) = 2^{-n-2}\frac{(n+1)!}{n_0!n_1!}\int_0^1 \frac{dt}{\sqrt{t}}\,(1-\sqrt{t})^{n_1}(1+\sqrt{t})^{n_0}\,. \tag{20}$$

This expression can be rewritten as

$$I_{\frac{1}{2}}(n_1+1; n_0+1) = 2^{-n-2}\frac{(n+1)!}{n_0!n_1!}\int_0^1 \frac{dt}{\sqrt{t}}\,(1-t)^{n_0}(1-\sqrt{t})^{\Delta}\,. \tag{21}$$

The last factor in the integrand can be expanded as the finite binomial series

$$(1-\sqrt{t})^{\Delta} = \sum_{m=0}^{\Delta}\frac{\Delta!}{m!\,(\Delta-m)!}(-1)^m\, t^{\frac{m}{2}}\,,$$

and this allows the evaluation of the integral, term for term:

$$I_{\frac{1}{2}}(n_1+1; n_0+1) = 2^{-n-2}\frac{(n+1)!}{n_0!\,n_1!}\sum_{m=0}^{\Delta}\frac{\Delta!}{m!\,(\Delta-m)!}(-1)^m\frac{\Gamma(\frac{m+1}{2})\Gamma(n_0+1)}{\Gamma(n_0+\frac{m+3}{2})}\,. \tag{22}$$

The Stirling expansion, namely

$$\Gamma(n) = (n-1)! = \sqrt{2\pi}n^{n-\frac{1}{2}}\,\mathrm{e}^{-n}[1+O(\tfrac{1}{n})]$$

is now used to give the asymptotic expressions

$$\begin{aligned}\frac{(n+1)!}{n_0!\,n_1!} &\sim 2^{n+1}\sqrt{\frac{n}{2\pi}}\\ \frac{\Gamma(n_0+1)}{\Gamma(n_0+\frac{m+3}{2})} &\sim n_0^{-\frac{m+1}{2}} = \left(\frac{2}{n-\Delta}\right)^{\frac{m+1}{2}}.\end{aligned}$$

On inserting these forms into Equation (22), we find that $I_{\frac{1}{2}}(n_1+1;n_0+1)$ is asymptotically equivalent to

$$\frac{1}{2\sqrt{\pi}}\sqrt{\frac{n}{n-\Delta}}\left[\Gamma(1/2)-\Delta\Gamma(1)\sqrt{\frac{2}{n-\Delta}}+\ldots+(-1)^{\Delta}\Gamma(\tfrac{\Delta+1}{2})\left(\frac{2}{n-\Delta}\right)^{\frac{\Delta}{2}}\right]$$

All the terms in the square braces, except for the first one, vanish in the limit of large n, and there is only a finite number of these terms. So only the first term survives, and since $\Gamma(1/2)=\sqrt{\pi}$, we have thereby proved indeed that $I_{\frac{1}{2}}(n_1+1;n_0+1)$ tends to $1/2$ in the limit as n tends to infinity.

Theorem 2
If $\Delta \sim n^{\beta}$ for large n, with $1/2 < \beta \leq 1$, then $I_{\frac{1}{2}}(n_1+1;n_0+1)$ tends to 0 in the limit.

Proof
Recall that we can rewrite Equation (19) as Equation (20). The latter can also be rewritten as

$$I_{\frac{1}{2}}(n_1+1;n_0+1) = 2^{-n-2}\frac{(n+1)!}{n_0!n_1!}\int_0^1 \frac{dt}{\sqrt{t}}(1-t)^{n_1}(1+\sqrt{t})^{-\Delta}, \qquad (23)$$

and we now split the integral into two pieces, corresponding to $0 < t < \varepsilon^2$ and $\varepsilon^2 < t < 1$, where ε will be specified in a moment. Clearly,

$$\int_0^{\varepsilon^2}\frac{dt}{\sqrt{t}}(1-t)^{n_1}(1+\sqrt{t})^{-\Delta} < \int_0^{\varepsilon^2}\frac{dt}{\sqrt{t}} = 2\varepsilon$$

whereas

$$\begin{aligned}\int_{\varepsilon^2}^1 \frac{dt}{\sqrt{t}}(1-t)^{n_1}(1+\sqrt{t})^{-\Delta} &< (1+\varepsilon)^{-\Delta}\int_0^1\frac{dt}{\sqrt{t}}(1-t)^{n_1}\\ &= (1+\varepsilon)^{-\Delta}\frac{\Gamma(\frac{1}{2})\Gamma(n_1+1)}{\Gamma(n_1+\frac{3}{2})}.\end{aligned}$$

We insert the last two inequalities into Equation (23) and also employ the Stirling expansion, as in the proof of Theorem 1. This yields

$$I_{\frac{1}{2}}(n_1+1;n_0+1) < \sqrt{\frac{n}{2\pi}}\varepsilon + \sqrt{\frac{n}{n+\Delta}}(1+\varepsilon)^{-\Delta}\,.$$

Now choose $\varepsilon = n^{-\alpha}$ and put $\Delta \sim n^{\beta}$, thereby obtaining

$$I_{\frac{1}{2}}(n_1+1;n_0+1) < \frac{1}{\sqrt{2\pi}}\,n^{\frac{1}{2}-\alpha} + (1+n^{-\alpha})^{-n^{\beta}}\,.$$

Now $(1+n^{-\alpha})^{n^{\alpha}}$ tends to e in the limit of large n, so we obtain

$$I_{\frac{1}{2}}(n_1+1;n_0+1) < \frac{1}{\sqrt{2\pi}}\,n^{\frac{1}{2}-\alpha} + \exp[-n^{\beta-\alpha}]$$

asymptotically. For any $\alpha > 1/2$, the first term above vanishes asymptotically, and for any $\beta > \alpha$, so does the second term. Hence for any $1/2 < \beta \leq 1$ we have shown that $I_{\frac{1}{2}}(n_1+1;n_0+1)$ tends to 0 in the limit as n tends to infinity.

Corollary
If $f = \frac{\Delta}{n}$ is constant, then $I_{\frac{1}{2}}(n_1+1;n_0+1)$ tends to 0 in the limit that $n = n_1 + n_0$ tends to infinity.

Proof
This follows immediately by taking the special case $\beta = 1$ in Theorem 2.

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