university of groningen

NWO
Netherlands Organisation for Scientific Research

Games and Decision Lunchtime meeting
5 October 2011

# Learning Juror Competence:
# a generalised Condorcet Jury Theorem

Jan-Willem Romeijn and David Atkinson
University of Groningen

# Contents

# 1 The Condorcet formula

A jury of $n$ members is trying Jack for murder. A number $n_1$ vote that Jack is guilty, $H^1$, while the remaining $n_0$ members vote that he is innocent, $H^0$. Jurors are characterised by

$$P(V_i^j|H^j \cap V_{i'}^{j'}) = P(V_i^j|H^j) = q_j > \frac{1}{2}.$$

If $H^j$ is in fact true, the event that jury member $i$ votes for $H^j$, denoted $V_i^j$, has some fixed chance $q_j$, the competence. We assume that the competences are greater than one half.

**Condorcet jury theorem**

We can now introduce Condorcet's jury theorem. Say that $H^1$ is true. For an ever larger jury size $n$, consider the relative frequency of voters in favour,

$$f_1 = \frac{n_1}{n} = 1 - f_0.$$

By the law of large numbers, the probability that $f_j$ differs from $q_j$ tends to 0. Because $q_j > \frac{1}{2}$, we have:

$$\lim_{n \to \infty} P\left(f_j > \frac{1}{2} \mid H^j\right) = 1.$$

This is a straightforward consequence of the probability of votes under the assumption of $H^j$.

## Inverse Condorcet theorem

Rather than calculating the probability of a majority of votes given the truth of $H^j$, we might ask for the probability of the hypothesis $H^j$ given some majority of votes:

$$P(H^1|V_{n\Delta}) = P(H^1) \frac{q_1^{n_1}(1-q_1)^{n_0}}{P(V_{n\Delta})}.$$

Here $V_{n\Delta} = \bigcap_{i=1}^{n} V_i^{u(i)}$ is the jury vote, $u(i)$ the vote of juror $i$, and $\Delta = n_1 - n_0$. With this we can derive an inverse version of Condorcet's theorem. If in the limit $f_j > 1/2$, then

$$\lim_{n\to\infty} P(H^j|V_{n\Delta}) = 1.$$

**Condorcet formula**

Under the idealising assumptions that

- the priors of $H^0$ and $H^1$ are equal, $P(H^1) = P(H^0)$, and that

- the competences of jury members on $H^0$ and $H^1$ are equal, $q_0 = q_1 = q$,

we have the following posterior odds:

$$\frac{P(H^1|V_{n\Delta})}{P(H^0|V_{n\Delta})} = \left(\frac{q}{1-q}\right)^{\Delta}.$$

It depends only on the absolute margin of the jury vote, and not on the number of jurors.

# 2   Counterintuitive consequences

List (2004) emphasises the significance of the absolute margin for jury votes. But the sole dependence on Δ is rather puzzling. Which of the following two juries do you prefer?

|                          | Small jury | Large jury |
| ------------------------ | :--------: | :--------: |
| Number of jurors         |     10     |    100     |
| Number in favour ($n_1$) |     10     |     56     |
| Number against ($n_0$)   |      0     |     44     |
| Absolute margin (Δ)      |     10     |     12     |

## A classical statistical analysis

A 95% confidence interval for juror competence $q$ shows that the votes are a freak accident.
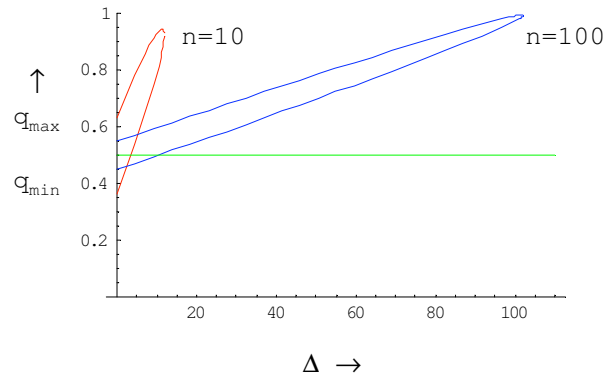


Alternatively, the jurors from the smaller jury are more competent.

**Responses in the literature**

There are some discussions of this aspect of jury votes.

- List (2004) drops symmetric competence so that the posterior odds come to depend on jury size, but not in the relevant way.

- Bovens and Hartmann (2004) argue that the coherence of jurors indicates the veracity of the jury verdict, and model this by introducing a positive correlation among jury votes.

- Goodin and Estlund (2004) propose to use an estimation of the competence, $\hat{q}_j = \frac{n_j}{n}$, in computing the posterior odds but this fails to express our confidence in the estimate.

We agree that the jury vote reveals the competence of the jury and aim to model this more precisely.

# 3   A model for learning competences

We split the hypotheses $H^0$ and $H^1$ up into $H_{q_0}$ and $H_{q_1}$ respectively. The hypotheses $H^j$ each consist of a range of statistical hypotheses:

$$P(H^0) = \int_{1/2}^{1} P(H_{q_0})\,dq_0 \qquad P(H^1) = \int_{1/2}^{1} P(H_{q_1})\,dq_1\,.$$

The hypotheses $H_{q_j}$ fix the competences to $P(V_j^i|H_{q_j} \cap V_{j'}^{i'}) = q_j$. We assume that the priors are equal and uniform over $(1/2, 1)$ for both $q_0$ and $q_1$.

## Transforming the problem

We can turn this into a well-known statistical problem by a suitable merger of the parameters $q_j$ into a single $r \in [0, 1]$, namely $q_0 = r$ and $q_1 = 1 - r$.
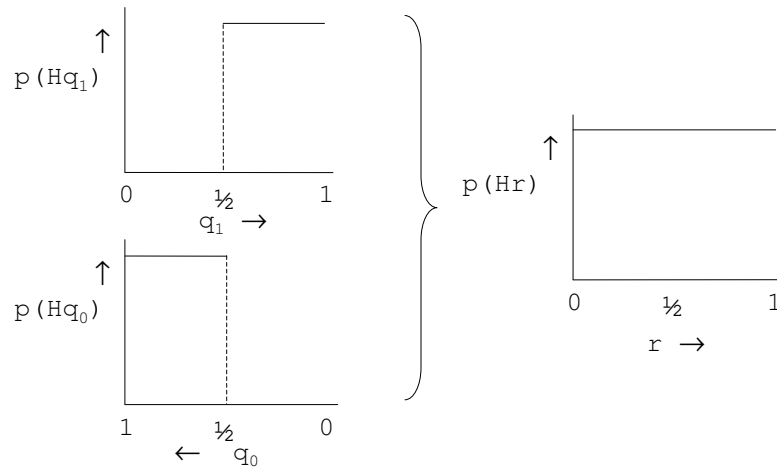
**Posterior for the hypothesis**

We model the impact of the jury vote on the probability assignments over $q_0$ and $q_1$ by modelling its impact on the probability assignment over $r$. The posterior over $H_r$ is a Beta distribution,

$$P(H_r|V_{n\Delta}) = \frac{(n+1)!}{n_0!n_1!}r^{n_1}(1-r)^{n_0}.$$

So the posterior probability of the hypothesis $H^0$ is:

$$P(H^0|V_{n\Delta}) = \frac{(n+1)!}{n_0!n_1!}\int_0^{1/2} r^{n_1}(1-r)^{n_0}\, dr.$$

# 4   Analytic and numerical results

We retain an important consequence of the Condorcet formula.  On the assumption that $\Delta = n_1 - n_0 > 0$, we have

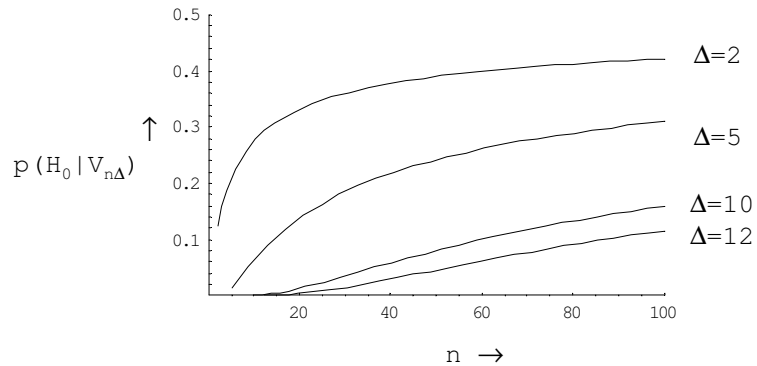$$P(H^1|V_{n\Delta}) > P(H^0|V_{n\Delta}).$$

But we can also show that

$$\frac{P(H^1|V_{n+2,\Delta})}{P(H^0|V_{n+2,\Delta})} < \frac{P(H^1|V_{n\Delta})}{P(H^0|V_{n\Delta})}.$$

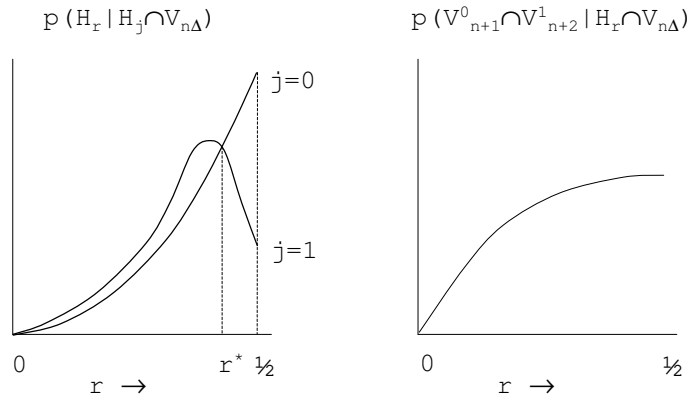This repairs the counterintuitive choice between the two juries.

## Dependence on jury size

These results are in accordance with the aforementioned intuitions on the relation between jury votes and the hypothesis voted over.
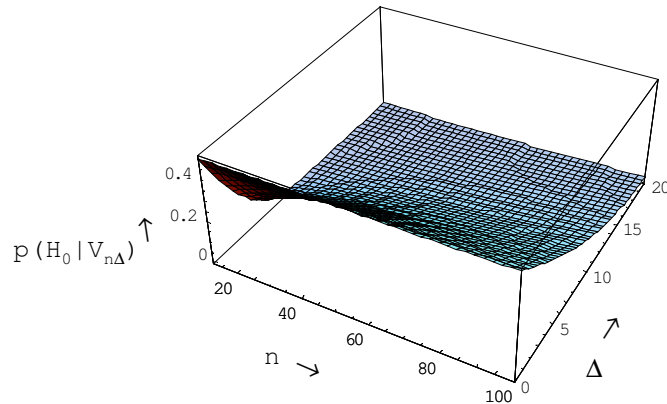
## Proof of monotonic dependence on jury size

Given the likelihoods $r(1-r)$ for $H_r$, the marginal likelihood of the hypothesis $H^0$ is larger because most of the mass lies close to $r = \frac{1}{2}$.



$p\,(H_r \,|\, H_j \cap V_{n\Delta})$

j=0

j=1

0

$r^\star$  ½

r  →

$p\,(V^0_{n+1} \cap V^1_{n+2} \,|\, H_r \cap V_{n\Delta})$

0

½

r  →

## Dependence on jury size and margin

For fixed jury size $n$, the probability of $H^0$ decreases with increasing majority size $\Delta$.



And for fixed $\Delta$ and increasing $n$, the probability of $H^0$ increases towards ½.

**Limiting behaviour**

For constant Δ, we find the asymptotic behaviour

$$\lim_{n\to\infty} P(H^0|V_{n\Delta}) = \frac{1}{2}.$$

For constant fractional majority, $f = \Delta/n > 0$, we have

$$\lim_{n\to\infty} P(H^0|V_{n,nf}) = 0.$$

In fact the increase in Δ need not be linear. It is enough if $\Delta \sim n^\beta$ for $\beta > \frac{1}{2}$.

# 5  Conclusions

In the model with competence learning we have:

- The probability that the jury majority verdict is incorrect is monotonically increasing in the jury size $n$, if the absolute margin $\Delta$ remains constant.

- The probability that the jury majority verdict is incorrect tends to one-half as $n$ tends to infinity, if $\Delta$ remains constant in this limit.

- The probability that the jury majority verdict is incorrect tends to zero as $n$ tends to infinity, if the fractional majority, $f = \Delta/n$, tends to a nonzero constant in this limit.

**Consequences**

For the discussion on voting rules, two consequences of this must be given extra emphasis.

- The exclusive dependence on the absolute margin is an artefact of idealising assumptions, not something inherent to jury verdicts.

- Both the normal Condorcet jury theorem and the converse Condorcet jury theorem for posterior odds remain valid in the new model.

This vindicates the intuition that the relative margin indicates the support provided by a jury vote.

**Further research**

Some suggestions on how to develop the results of the present paper.

- In the interest of practical applicability, we need to relax the assumption on the independence of the jurors.

- Variation of competences within the jury can be captured by hierarchical models.

- We can apply the foregoing to the discussion over coherence measures in epistemology.

# Thanks!

This talk and the paper on which it is based are both available at

   http://www.philos.rug.nl/˜romeyn.

For comments and questions, email j.w.romeijn@rug.nl.